A divisibility problem

Question

Prove that $2^{3n} - 1$ is divisible by 7 , where n is a natural number .

Method 1 Mathematical Induction

Let $P(n)$ be the proposition : $2^{3n} - 1 = 7a_n$, where a_n is a natural number .

For $P(n)$, $2^{3(1)} - 1 = 8 - 1 = 7 = 7a_1$, where $a_1 = 1$ is a natural number .

Assume $P(k)$ is true for some natural number k , that is,

$$2^{3k} - 1 = 7a_k \text{ , where } a_k \text{ is a natural number} \quad \ldots\ (1)$$

For $P(k + 1)$,

$$2^{3(k+1)} - 1 = 2^{3k+3} - 1 = 8(2^{3k}) - 1 = 8(2^{3k} - 1) + 7 = 8\,(7a_k\,) + 7 \text{ , by (1) .}$$

$$= 7(8a_k + 1) = 7a_{k+1} \text{ , where } a_{k+1} = 8a_k + 1 \text{ is a natural number .}$$

$\therefore$ $P(k + 1)$ is true .

$\therefore$ By the Principle of Mathematical Induction, $P(n)$ is true for all natural numbers n .

$\therefore$ $2^{3n} - 1$ is divisible by 7 , where n is a natural number .

Method 2 Binominal Theorem

$$2^{3n} - 1 = 8^n - 1 = (7 + 1)^n - 1$$

$$= \left[7^n + C_1^n 7^{n-1}(1) + C_2^n 7^{n-2}(1)^2 + \ldots + C_{n-1}^n 7(1)^{n-1} + 1^n\right] - 1 \text{ , by Binominal Theorem .}$$

$$= 7^n + C_1^n 7^{n-1} + C_2^n 7^{n-2} + \ldots + C_{n-1}^n 7$$

$$= 7\left(7^{n-1} + C_1^n 7^{n-2} + C_2^n 7^{n-3} + \ldots + C_{n-1}^n\right)$$

Since C_r^n is a natural number, $2^{3n} - 1$ is obviously divisible by 7 .

Method 3 Factor Theorem

Let $f(n) = x^n - a^n$, where n is a natural number .

Since $f(a) = a^n - a^n = 0$

$\therefore$ By Factor Theorem, $x^n - a^n$ is divisible by $(x - a)$.

Putting $x = 8$, $a = 1$, we have :

$8^n - 1^n$ is divisible by $(8 - 1)$

$\therefore$ $2^{3n} - 1$ is divisible by 7 , where n is a natural number .

Note : $x^n - a^n = (x - a)\,(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + xa^{n-2} + a^{n-1})$

Hence $2^{3n} - 1 = 7(8^{n-1} + 8^{n-2} + \ldots + 8 + 1)$

Method 4 **Geometric Series**

For geometric series, we have the formula:

$S(n) = a\frac{r^n - 1}{r - 1}$, where a is the first term, r is the common ratio of Geom. series

$\therefore \quad 1 + 8 + 8^2 + 8^3 + \ldots + 8^{n-1} = 1 \times \frac{8^n - 1}{8 - 1}$

Hence, $2^{3n} - 1 = 7(8^{n-1} + 8^{n-2} + \ldots + 8 + 1)$

Method 5 **Binary number**

$(2^3 - 1)_{(10)} = 111_{(2)}$

$(2^6 - 1)_{(10)} = 111{,}111_{(2)}$

Hence, we have $(2^{3n} - 1)_{(10)} = 111{,}111{,}\ldots{,}111_{(2)}$ (there are $3n$ number of 1 or n groups of 111)

Also, $7_{(10)} = 111_{(2)}$.

$\therefore \quad (2^{3n} - 1)_{(10)} / 7_{(10)} = 111{,}111{,}\ldots{,}111_{(2)} / 111_{(2)}$

$= 1{,}001, \ldots, 001_{(2)}$

$\therefore \quad 2^{3n} - 1$ is divisible by 7 , where n is a natural number .

Method 6 **Base 8**

Base 8 gives an even simpler proof . According to the definition of base 8,

$(8^n)_{(10)} = 100\ldots000_{(8)}$ (there are n zeros)

$(8^n - 1)_{(10)} = 77\ldots77_{(8)}$ (there are n sevens)

$(2^{3n} - 1)_{(10)} / 7_{(10)} = 77\ldots77_{(8)} / 7_{(8)} = 11\ldots11_{(8)}$

$\therefore \quad 2^{3n} - 1$ is divisible by 7 , where n is a natural number .

Method 7 **Modular arithmetic**

If a, b, n are integers, $a - b$ is an integer multiple of n , we write $a \equiv b \pmod{n}$.

For those who know modular arithmetic, the proof is simple:

Observe that $8 \equiv 1 \pmod{7} \Rightarrow 8^n \equiv 1^n \pmod{7} \equiv 1 \pmod{7}$

$\therefore \quad 2^{3n} \equiv 1 \pmod{7}$

$\therefore \quad 2^{3n} - 1$ is divisible by 7 , where n is a natural number .